Explode-Decay Solitons in the Generalized Inhomogeneous Higher-Order Nonlinear Schrödinger Equations

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In this paper we investigate the generalized inhomogeneous higher-order nonlinear Schrödinger equations, generated recently by deforming the inhomogeneous Heisenberg ferromagnetic spin system through a space curve formalism [Phys. Lett. A **352**, 64 (2006)] and construct their multisoliton solutions, using gauge transformation. The amplitude of the bright soliton solutions generated grows and decays with time, and there is an exchange of energy between soliton trains during interaction. – PACS numbers: 02.30.lk, 02.30.Jr, 05.45.Yv.

Key words: Generalized Inhomogeneous Higher-Order Nonlinear Schrödinger Equation; Gauge Transformation; Explode-Decay Solitons.

1. Introduction

It is well known that integrable nonlinear partial differential equations (PDEs) in (1+1)-dimensions, solvable by the inverse scattering transform, can be associated with the motion of a nonlinear string of constant length or a space curve with an appropriate equation of motion in the Euclidean space E^3 [1 – 4]. This interpretation of integrable models stimulated much interest in the study of nonlinear dynamical systems, leading to their investigation from the viewpoint of differential geometry. In fact, this one-to-one correspondence between the integrable nonlinear PDEs and the motion of a space curve paved the way for the mapping of the isotropic Heisenberg ferromagnetic (HF) spin chain in the continuum limit with the nonlinear Schrödinger (NLS) equation [5, 6]. Incidentally, this led to the opening of the floodgates between magnetic spin systems and the celebrated integrable models.

It should be mentioned that these integrable equations with constant coefficients are regarded to be highly idealized in physical situations. Hence it was believed that a realistic description of physical phenomena around us should take into account the inhomogeneities/nonuniformities in the medium. In fact, there was a spurt in the study of wave propagation through inhomogeneous media and the associated variable coefficient nonlinear PDEs eversince the identification of solitons in them [7, 8]. These variable coefficient integrable equations have a wide range of applications in the propagation of radio waves in the ionosphere, waves in the ocean, optical pulses in glass fibres, laser radiations in a plasma and impurities in magnetic systems [9, 10]. The observation of solitons in nonuniform optical fibers [11] underscored a thorough investigation of such systems from the viewpoint of technology. In the above nonlinear PDEs, the spectral parameter is regarded as a variable quantity that satisfies in general an overdetermined system of PDEs which is uniquely determined by the auxiliary linear eigenvalue problem.

Burtsev et al. [12] have generated the deformations of various well known integrable equations and have also proposed that to every equation with a constant spectral parameter to which the scheme of inverse scattering transform [13] is applicable, there corresponds an entire class of equations with a variable spectral parameter. In addition, the space curve formalism has also been employed to generate these inhomogeneous nonlinear PDEs [14]. These equations were shown to be completely integrable possessing Lax pair, Bäcklund transformation, infinite number of conservation laws, soliton solutions etc. [8, 15].

In this paper we investigate the inhomogeneous NLS equation generated recently by deforming the inhomogeneous Heisenberg ferromagnetic spin system through the space curve formalism, using the prolongation structure theory [16], and generate the bright soliton solutions from the associated linear eigenvalue problems, using gauge transformation. We have also brought out the interaction of solitons.

2. Higher-Order Inhomogeneous Nonlinear Schrödinger (NLS) Equations and Multisoliton Solutions

The inhomogeneous fourth order NLS equation, generated recently using the prolongation structure theory of Wahlquist and Estabrook [1], has the form

$$\begin{aligned} &\mathrm{i}q_t + \varepsilon q_{xxxx} + 8\varepsilon|q|^2 q_{xx} + 2\varepsilon q^2 q^*_{xx} + 4\varepsilon|q_x|^2 q \\ &\quad + 6\varepsilon q^* q_x^2 + 6\varepsilon|q|^4 q + (fq)_{xx} \\ &\quad + 2q\left\{ f|q|^2 + \int_{-\infty}^{x} \mathrm{d}x' f_x |q|^2 \right\} - \mathrm{i}(hq)_x = 0, \end{aligned} \tag{1}$$

where $q = q(x,t)$ is a complex variable, and f and h represent inhomogeneities present in the medium and are linear functions of the spatial variable x of the form

$$f = \mu_1 x + \nu_1, \quad h = \mu_2 x + \nu_2, \tag{2}$$

and ε, μ_1, μ_2, ν_1, and ν_2 are parameters.

Even though the Painlevé property of (1) has not been investigated, it is expected to be completely integrable and admits the following linear eigenvalue problem:

$$\Phi_x = \begin{pmatrix} -\mathrm{i}\lambda & q \\ -q^* & \mathrm{i}\lambda \end{pmatrix} \Phi = U\Phi, \tag{3a}$$

$$\Phi_t = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix} \Phi = V\Phi, \tag{3b}$$

where λ is a complex spectral parameter in the system, and the components A and B take the following form:

$$\begin{aligned} A = {} & \mathrm{i}\varepsilon(q^* q_{xx} + q q^*_{xx} - |q_x|^2) + 3\mathrm{i}\varepsilon|q|^4 \\ & - 2\lambda\varepsilon(q^* q_x - q q^*_x) - 4\mathrm{i}\lambda^2\varepsilon|q|^2 + 8\mathrm{i}\varepsilon\lambda^4 \\ & + \mathrm{i}\int_{-\infty}^{x} \mathrm{d}x' f_x |q|^2 + \mathrm{i}f|q|^2 - 2\mathrm{i}f\lambda^2 - \mathrm{i}h\lambda, \end{aligned} \tag{4a}$$

$$\begin{aligned} B = {} & \mathrm{i}\varepsilon q_{xxx} + 2\varepsilon\lambda q_{xx} + 6\mathrm{i}\varepsilon|q|^2 q_x - 4\mathrm{i}\varepsilon\lambda^2 q_x \\ & + 4\varepsilon\lambda|q|^2 q - 8\varepsilon\lambda^3 q + \mathrm{i}(fq)_x + 2f\lambda q + hq. \end{aligned} \tag{4b}$$

In the above eigenvalue problem the spectral parameter λ is nonisospectral obeying the equation

$$\lambda_t = 2\mu_1\lambda^2 + \mu_2\lambda. \tag{5}$$

It is obvious that the compatibility condition $U_t - V_x + [U,V] = 0$ generates (1). Now, to generate the soliton solutions of (1) [17] using gauge transformation, we consider the seed solution $q^{(0)} = 0$ to give the vacuum linear systems

$$\Phi^{(0)}_x = \begin{pmatrix} -\mathrm{i}\lambda & 0 \\ 0 & \mathrm{i}\lambda \end{pmatrix} \Phi^{(0)} = U^{(0)}\Phi^{(0)}, \quad \Phi^{(0)}_t = \begin{pmatrix} 8\mathrm{i}\varepsilon\lambda^4 - 2\mathrm{i}f\lambda^2 - \mathrm{i}h\lambda & 0 \\ 0 & -8\mathrm{i}\varepsilon\lambda^4 + 2\mathrm{i}f\lambda^2 + \mathrm{i}h\lambda \end{pmatrix} \Phi^{(0)} = V^{(0)}\Phi^{(0)}. \tag{6}$$

Solving the above vacuum linear systems, keeping in mind the variation of the spectral parameter with t by virtue of (5), we have

$$\Phi^{(0)}(x,t,\lambda) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\lambda x + \int (8\mathrm{i}\varepsilon\lambda^4 - \mathrm{i}(2\nu_1\lambda^2 + \nu_2\lambda))\mathrm{d}t} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\lambda x + \int(-8\mathrm{i}\varepsilon\lambda^4 + \mathrm{i}(2\nu_1\lambda^2 + \nu_2\lambda))\mathrm{d}t} \end{pmatrix}. \tag{7}$$

Now, effecting a gauge transformation

$$\Phi^{(1)}(x,t,\lambda) = \chi_1 \Phi^{(0)}(x,t,\lambda), \tag{8}$$

the new eigenvalue problem takes the form

$$\Phi^{(1)}_x = U^1\Phi^{(1)}, \quad \Phi^{(1)}_t = V^1\Phi^{(1)} \tag{9}$$

with

$$U^{(1)} = \chi_1 U^{(0)} \chi_1^{-1} + \chi_{1x}\chi_1^{-1}, \tag{10a}$$

$$V^{(1)} = \chi_1 V^{(0)} \chi_1^{-1} + \chi_{1t}\chi_1^{-1}. \tag{10b}$$

Choosing χ_1 as a meromorphic function of the associated Riemann problem, we have

$$\chi_1 = \left(1 + \frac{\lambda_1 - \zeta_1}{\lambda - \lambda_1} P_1(x,t)\right) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{11}$$

where λ_1 and ζ_1 are arbitrary complex parameters and P_1 is a projection matrix ($P_1^2 = P_1$). Imposing the constraint that U^1 and V^1 do not develop singularities around $\lambda = \lambda_1$ and $\lambda = \zeta_1$, the choice of the projection matrix is governed by the following set of PDEs:

$$\begin{aligned} P_{1x} = {} & (1 - P_1)\sigma_3 U^{(0)}(\zeta_1)\sigma_3 P_1 \\ & - P_1\sigma_3 U^{(0)}(\lambda_1)\sigma_3(1 - P_1), \end{aligned} \tag{12a}$$

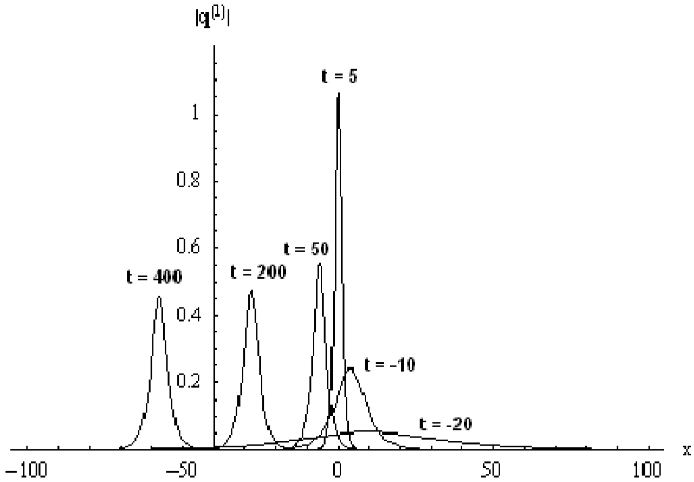

Fig. 1. Profile of the bright explode-decay soliton at various time intervals for the parametric choice: $\mu_1 = 0.1$; $\mu_2 = 0.05$; $\varepsilon = 0.01$; $\nu_1 = 0.1$; $\nu_2 = 0.2$; $\delta_1 = 0.1$; $\phi_1 = 0.1$.

$$P_{1t} = (1-P_1)\sigma_3 V^{(0)}(\zeta_1)\sigma_3 P_1 - P_1\sigma_3 V^{(0)}(\lambda_1)\sigma_3(1-P_1), \tag{12b}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{12c}$$

The above system of equations suggests that P_1 depends only on the trivial matrix eigenfunction $\Phi^{(0)}(x,\lambda)$, a diagonal matrix, and has a compact form given by

$$P_1(x,t) = \sigma_3 \frac{M^{(1)}}{[\text{trace}\, M^{(1)}]}\sigma_3, \tag{13a}$$

$$M^{(1)} = \Phi^{(0)}(x,t,\zeta_1)\begin{pmatrix} m_1 & 1/n_1 \\ n_1 & 1/m_1 \end{pmatrix}\Phi^{(0)}(x,t,\lambda_1)^{-1}, \tag{13b}$$

where m_1 and n_1 are arbitrary complex constants.

Hence, solving the above system of equations by choosing $\lambda_1 = \alpha_1(t) + \mathrm{i}\beta_1(t)$, $\zeta_1 = \lambda_1^*$, we get

$$P_1(x,t) = \begin{pmatrix} \frac{1}{2}\mathrm{sech}[\theta_1]\exp[\theta_1] & -\frac{1}{2}\mathrm{sech}[\theta_1]\exp[\mathrm{i}\xi_1] \\ -\frac{1}{2}\mathrm{sech}[\theta_1]\exp[-\mathrm{i}\xi_1] & \frac{1}{2}\mathrm{sech}[\theta_1]\exp[-\theta_1] \end{pmatrix}, \tag{14}$$

where

$$\theta_1 = -2\beta_1 x + \int (16\varepsilon(4\alpha_1^3\beta_1 - 4\alpha_1\beta_1^3) - 8\nu_1\alpha_1\beta_1 - 2\nu_2\beta_1)\mathrm{d}t + 2\delta_1, \tag{15a}$$

$$\xi_1 = -2\alpha_1 x + \int (16\varepsilon(\alpha_1^4 + \beta_1^4 - 6\alpha_1^2\beta_1^2) - 4\nu_1(\alpha_1^2 - \beta_1^2) - 2\nu_2\alpha_1)\mathrm{d}t - 2\phi_1, \tag{15b}$$

$$\alpha_{1t} = 2\mu_1(\alpha_1^2 - \beta_1^2) + \mu_2\alpha_1, \tag{15c}$$

$$\beta_{1t} = 4\mu_1\alpha_1\beta_1 + \mu_2\beta_1, \tag{15d}$$

where δ_1, ϕ_1 are arbitrary real constants.

Using (6a), (11) and (12a) in (10a), we obtain the bright soliton solution as

$$q^{(1)} = -q^{(0)} - 2\mathrm{i}(\lambda_1 - \zeta_1)[\sigma_3 P_1 \sigma_3]_{12} = 2\beta_1 \mathrm{sech}[\theta_1]\exp[\mathrm{i}\xi_1]. \tag{16}$$

Looking at the nature of the bright soliton solution, it is evident that its amplitude, which depends on β_1, is time-dependent by virtue of (15d), and hence it oscillates with time. In other words, the amplitude of the bright soliton grows and decays with time depending on the parameters μ_1 and μ_2 and the initial conditions required for solving the two ordinary differential equations (15c) and (15d). Hence one calls such solutions as "explode-decay solitons", unlike the solitons in the integrable homogeneous NLS equation. It is also evident from (15) and (16) that velocities of these explode-decay solitons, which depend on α_1 and β_1, also vary with time. The profile of the bright soliton shown in Fig. 1 confirms this observation. This process can be continued further to generate multisoliton solutions. For example, to construct a two-soliton solution, we take the one soliton given by (16) as the seed solution and gauge transform $\Phi^{(1)}$ by a wave function χ_2 to give the following systems:

$$\Phi_x^{(2)} = U^{(2)}\Phi^{(2)}, \quad \Phi_t^{(2)} = V^{(2)}\Phi^{(2)}, \tag{17}$$

where

$$\Phi^{(2)} = \chi_2\Phi^{(1)}, \tag{18a}$$

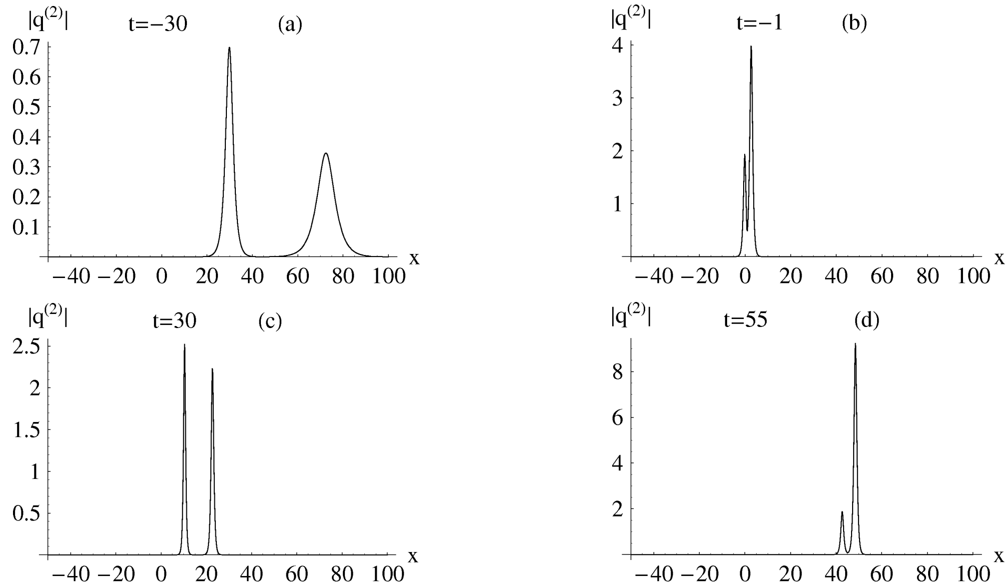

Fig. 2. Two-soliton interaction in the deformed NLS equation (1) for the parametric choice: $\mu_1 = 0.01$; $\mu_2 = 0.02$; $\varepsilon = 0.001$; $\nu_1 = 0.5$; $\nu_2 = 0.25$; $\delta_1 = 0.001$; $\phi_1 = 0.001$; $\delta_2 = 0.002$; $\phi_2 = 0.002$.

$$U^{(2)} = \chi_2 U^{(1)} \chi_2^{-1} + \chi_{2x}\chi_2^{-1}, \tag{18b}$$

$$V^{(2)} = \chi_2 V^{(1)} \chi_2^{-1} + \chi_{2t}\chi_2^{-1}, \tag{18c}$$

and the meromorphic function χ_2 assumes the same form as (11), i. e.

$$\chi_2 = \left(1 + \frac{\lambda_2 - \zeta_2}{\lambda - \lambda_2} P_2(x,t)\right)\sigma_3, \tag{19}$$

and the projection matrix P_2 again satisfies the system of equations similar to (12), except that $U^{(0)}$ and $V^{(0)}$ are replaced by $U^{(1)}$ and $V^{(1)}$ and λ_2 and ζ_2 replace λ_1 and ζ_1. Thus, the explicit form of the two-soliton solution can be given by

$$q^{(2)} = -q^{(1)} - 2\mathrm{i}(\lambda_2 - \zeta_2)[\sigma_3 P_2 \sigma_3]_{12}, \tag{20}$$

where

$$P_2(x,t) = \sigma_3 \frac{M^{(2)}}{[\text{trace } M^{(2)}]}\sigma_3, \tag{21a}$$

$$M^{(2)} = \Phi^{(0)}(x,t,\zeta_2)\begin{pmatrix} m_2 & 1/n_2 \\ n_2 & 1/m_2 \end{pmatrix}\Phi^{(0)}(x,t,\lambda_2)^{-1}. \tag{21b}$$

The two-soliton solution can now be rewritten as [using (20) and (21)]

$$q^{(2)} = \frac{A_1 + A_2 + A_3 + A_4}{B_1 + B_2}, \tag{22}$$

where the components assume the following form:

$$A_1 = \{-2\beta_2[(\alpha_2-\alpha_1)^2 - (\beta_1^2 - \beta_2^2)] - 4\mathrm{i}\beta_1\beta_2(\alpha_2-\alpha_1)\}\mathrm{e}^{(\theta_1+\mathrm{i}\xi_2)}, \tag{23a}$$

$$A_2 = -2\beta_2[(\alpha_2-\alpha_1)^2 + (\beta_1^2 + \beta_2^2)]\mathrm{e}^{(-\theta_1+\mathrm{i}\xi_2)}, \tag{23b}$$

$$A_3 = \{-2\beta_1[(\alpha_2-\alpha_1)^2 + (\beta_1^2 - \beta_2^2)] + 4\mathrm{i}\beta_1\beta_2(\alpha_2-\alpha_1)\}\mathrm{e}^{(\mathrm{i}\xi_1+\theta_2)}, \tag{23c}$$

$$A_4 = -4\mathrm{i}\beta_1\beta_2[(\alpha_2-\alpha_1) - \mathrm{i}(\beta_1-\beta_2)]\mathrm{e}^{(\mathrm{i}\xi_1-\theta_2)}, \tag{23d}$$

$$B_1 = -4\beta_1\beta_2[\sinh(\theta_1)\sinh(\theta_2) + \cos(\xi_1-\xi_2)], \tag{23e}$$

$$B_2 = 2\cosh(\theta_1)\cosh(\theta_2) \cdot [(\alpha_2-\alpha_1)^2 + (\beta_1^2+\beta_2^2)], \tag{23f}$$

and

$$\theta_j = -2\beta_j x + \int [64\varepsilon(\alpha_j^3\beta_j - \alpha_j\beta_j^3) - 8\nu_1\alpha_j\beta_j - 2\nu_2\beta_j]\,\mathrm{d}t + 2\delta_j, \tag{24a}$$

$$\xi_j = -2\alpha_j x + \int [16\varepsilon(\alpha_j^4 + \beta_j^4 - 6\alpha_j^2\beta_j^2) - 4\nu_1(\alpha_j^2 - \beta_j^2) - 2\nu_2\alpha_j]\,\mathrm{d}t - 2\phi_j, \tag{24b}$$

$$\alpha_{jt} = 2\mu_1(\alpha_j^2 - \beta_j^2) + \mu_2\alpha_j, \tag{24c}$$

$$\beta_{jt} = 4\mu_1\alpha_j\beta_j + \mu_2\beta_j, \quad j = 1,2. \tag{24d}$$

Figure 2 portrays the time evolution of the two-soliton solution. From this, we observe that the two-soliton trains exchange energy between them as they propagate along the positive x-direction. This type of inelastic collision of soliton trains stems from the non-isospectral nature of the spectral parameter [time evolution of the spectral parameter λ by virtue of (5)] and this can be attributed to the inhomogeneities present in the medium during wave propagation. It should be mentioned that, even though the shape of the soliton trains changes, their total energy is preserved. This can again be generalized to N-soliton solutions of the form

$$q^{(N)} = -q^{(N-1)} - 2\mathrm{i}(\lambda_N - \zeta_N)[\sigma_3 P_N \sigma_3]_{12}. \quad (25)$$

There exists another fifth order inhomogeneous NLS equation, generated again by deforming the inhomogeneous Heisenberg ferromagnetic system using the prolongation structure theory, and it has the following form:

$$\begin{aligned} &\mathrm{i}q_t - \mathrm{i}\varepsilon q_{xxxxx} - 10\mathrm{i}\varepsilon|q|^2 q_{xxx} - 20\mathrm{i}\varepsilon q_x q^* q_{xx} \\ &\quad - 30\mathrm{i}\varepsilon|q|^4 q_x - 10\mathrm{i}\varepsilon(|q_x|^2 q)_x + (fq)_{xx} \\ &\quad + 2q\left\{ f|q|^2 + \int_{-\infty}^{x} \mathrm{d}x' f_x |q|^2 \right\} - \mathrm{i}(hq)_x = 0. \end{aligned} \quad (26)$$

The Lax pair of the above system has the same form as (3) with

$$\begin{aligned} A = {} & \varepsilon(q^* q_{xxx} - q q^*_{xxx} + q_x q^*_{xx} - q_{xx} q^*_x \\ & + 6|q|^2 q^* q_x - 6|q|^2 q^*_x q) \\ & - 2\mathrm{i}\lambda\varepsilon(q q^*_{xx} + q^* q_{xx} - |q_x|^2 + 3|q|^4) \\ & + 4\lambda^2\varepsilon(q q^*_x - q_x q^*) + 8\mathrm{i}\lambda^3\varepsilon|q|^2 \\ & + \mathrm{i}\int_{-\infty}^{x} \mathrm{d}x' f_x |q|^2 + \mathrm{i}f|q|^2 - 2\mathrm{i}f\lambda^2 - \mathrm{i}h\lambda - 16\mathrm{i}\lambda^5\varepsilon, \end{aligned} \quad (27)$$

$$\begin{aligned} B = {} & \varepsilon(q_{xxxx} + 8|q|^2 q_{xx} + 2q^2 q^*_{xx} + 4|q_x|^2 q \\ & + 6q_x^2 q^* + 6|q|^4 q) - 2\mathrm{i}\lambda\varepsilon(q_{xxx} + 6|q|^2 q_x) \\ & - 4\lambda^2\varepsilon(q_{xx} + 2|q|^2 q) + 8\mathrm{i}\lambda^3\varepsilon q_x + 16\lambda^4\varepsilon q \\ & + \mathrm{i}(fq)_x + 2f\lambda q + hq, \end{aligned} \quad (28)$$

and the nonisospectral parameter λ again satisfies (5).

Soliton solutions for the above system can also be generated as before, using gauge transformation. For example, the bright soliton solution of the deformed NLS equation (26) is given by

$$q^{(1)} = 2\beta_1 \operatorname{sech}[\theta_1]\exp[\mathrm{i}\xi_1], \quad (29)$$

where

$$\begin{aligned} \theta_1 = {} & -2\beta_1 x + \int (32\varepsilon(-5\alpha_1^4\beta_1 + 10\alpha_1^2\beta_1^3 - \beta_1^5) \\ & - 8\nu_1\alpha_1\beta_1 - 2\nu_2\beta_1)\mathrm{d}t + 2\delta_1, \end{aligned} \quad (30a)$$

$$\begin{aligned} \xi_1 = {} & -2x\alpha_1 + \int (32\varepsilon(-\alpha_1^5 + 10\alpha_1^3\beta_1^2 - 5\alpha_1\beta_1^4) \\ & - 4\nu_1(\alpha_1^2 - \beta_1^2) - 2\nu_2\alpha_1)\mathrm{d}t - 2\phi_1, \end{aligned} \quad (30b)$$

$$\alpha_{1t} = 2\mu_1(\alpha_1^2 - \beta_1^2) + \mu_2\alpha_1, \quad (30c)$$

$$\beta_{1t} = 4\mu_1\alpha_1\beta_1 + \mu_2\beta_1. \quad (30d)$$

This can again be generalized to N-soliton solutions. Knowing the soliton solutions of the above inhomogeneous NLS-type equations, one can generate the soliton solutions of the associated inhomogeneous Heisenberg ferromagnetic spin systems through both geometrical and gauge equivalence.

3. Discussion

In this paper we have investigated the higher-order inhomogeneous NLS-type equations and generated their soliton solutions. We found that the amplitude of solitons grows and decays with time, and soliton trains exchange energy during propagation. The question, whether there exist other integrable deformations of inhomogeneous Heisenberg ferromagnetic spin systems remained open and is currently under investigation. Exploration of the Painlevé property of these systems for a more general f and h (other than a linear function of x) is also being analyzed, and the results will be published later.

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[1] H. D. Wahlquist and F. B. Estabrook, J. Math. Phys. **16**, 1 (1975); **17**, 1293 (1976).
[2] R. Herman, Phys. Rev. Lett. **36**, 835 (1976); **37**, 1591 (1976).
[3] H. C. Morris, J. Math. Phys. **78**, 533 (1977).
[4] R. K. Dodd and J. D. Gibbon, Proc. R. Soc. London **358**, 287 (1978).
[5] G. L. Lamb, Jr., Phys. Rev. Lett. **37**, 235 (1976); J. Math. Phys. **18**, 1654 (1977).
[6] M. Lakshmanan, Phys. Lett. A **64**, 353 (1978); J. Math. Phys. **20**, 1667 (1979).
[7] F. Calogero and A. Degasperis, Commun. Math Phys. **63**, 155 (1978).
[8] M. Lakshmanan and R. K. Bullough, Phys. Lett. A **80**, 287 (1980).
[9] F. Abdulleav, Theory of Soliton in Inhomogeneous Media, John Wiley and Sons, New York 1984.
[10] H. H. Chen and C. S. Liu, Phys. Fluids **21**, 377 (1978).
[11] G. P. Agarwal, Nonlinear Fiber Optics, Academic Press, New York 1989.
[12] S. P. Burtsev, V. E. Zakharov, and A. V. Mikhailov, Theor. Math. Phys. **70**, 323 (1987).
[13] V. E. Zakharov and A. B. Shabat, Fun. Anal. Appl. **8**, 228 (1974).
[14] M. Daniel, K. Porsezian, and M. Lakshmanan, J. Math. Phys. **35**, 6498 (1994).
[15] K. Porsezian, Pramana **48**, 143 (1997).
[16] W.-Z. Zhao, Y. Q. Bai, and K. Wu, Phys. Lett. A **352**, 64 (2006).
[17] L. L. Chau, J. C. Shaw, and H. C. Yen, J. Math. Phys. **32**, 1737 (1991).